INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS VOL. 21, 433–442 (1995)

SOLUTIONS OF MAGNETOHYDRODYNAMIC PROBLEMS BASED ON A CONVENTIONAL COMPUTATIONAL FLUID DYNAMICS CODE

S. S. SAZHIN AND M. MAKHLOUF

Fluent Europe Ltd, Hutton's Buildings, 146 West Street, Sheffield S1 4ES, U.K.

T. ISHII

Fukuama Reseach Department, Material and Processing Research Center, NKK Corporation, 1 Kokan-cho, Fukuyama city, Hiroshima-PRF, 721 Japan

SUMMARY

It is pointed out that there exists a hidden analogy between magnetohydrodynamic (MHD) and conventional computational fluid dynamic (CFD) equations. This allows the generalization of any conventional CFD code so that the effects of MHD can be accounted for. This generalization is actually made for the FLUENT CFD code. Although this generalized FLUENT code can easily be adjusted to any MHD environment, it has been specifically designed for metallurgical applications. Predictions of the code are validated against the analytical solutions for the Poiseuille–Hartmann flow and for the shielding of magnetic field oscillations by a conducting medium (skin effect).

KEY WORDS: computational fluid dynamics; magnetohydrodynamics

1. INTRODUCTION

Different approaches to MHD modelling have been discussed in numerous papers[1–7] and results of this modelling have been used in a number of industrial applications including metallurgical engineering and electromagnetic processing of materials.[8–12] Without going into details of numerical analyses of these papers we just note that their approach was based on the combined solution of MHD and CFD equations with MHD part of the code being in a certain sense independent of its CFD part. In contrast to these papers we will draw attention to the hidden analogy between the MHD equations and those of conventional CFD. This analogy makes it possible to generalize any conventional CFD code so that the effects of MHD can be accounted for. In our case this generalization has been applied to the FLUENT CFD code which has been widely used for different industrial applications (including such non-trivial applications as the solution of the Bolzmann equation[13] and modelling of the processes in CO_2 lasers[14]).

Basic equations of MHD and their approximations are discussed in Section 2. In Section 3 we discuss an analogy between MHD and conventional CFD equations and numerical aspects of the implementation of MHD equations into the FLUENT code. In Section 4 we discuss an application of the generalized FLUENT code to simple problems of the Poiseuille–Hartmann flow and the shielding

CCC 0271–2091/95/050433–10

Received November 1994
Revised March 1995

of magnetic field oscillations by a conducting medium, when the predictions of FLUENT can be verified against the analytical solutions. In the same section we briefly discuss the application of our code to MHD simulations used for modelling the continuous casting process. The main conclusions of the paper are summarized in Section 5.

2. BASIC EQUATIONS AND APPROXIMATIONS

Magnetic and electric fields do not influence the mass conservation equation, which can be written in exactly the same form as in conventional hydrodynamics. The momentum conservation (Navier–Stokes) equation, however, gets an extra electromagnetic force $\mathbf{j} \times \mathbf{B}$, where $\mathbf{j}$ is the electric current density, $\mathbf{B}$ is the magnetic field induction. As a result this equation in MHD can be written as

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F}_h + \mathbf{j} \times \mathbf{B}, \tag{1}$$

where ρ is the liquid or gas density, $\mathbf{v}$ is the velocity, $\mathbf{F}_h$ summarizes all forces in conventional hydrodynamics. In a similar way the enthalpy (h) equation in MHD can be written as

$$\frac{\mathrm{d}(\rho h)}{\mathrm{d}t} = Q_h + [\mathbf{j} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})] = Q_h + \mathbf{j}^2/\sigma, \tag{2}$$

where Q_h summarizes all terms used in the conventional hydrodynamics, $\mathbf{E}$ is the electrical field strength, $\mathbf{v}$ is the average mass velocity. Equation (2) effectively describes the heating of the medium by electrical current.

When the distributions of $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ are known then the MHD equations can be solved in exactly the same way as equations of conventional hydrodynamics with additional terms in equations (1) and (2). The complications arise due to the fact that the motion of gas or liquid in MHD influences the values of $\mathbf{j}$, $\mathbf{E}$ and $\mathbf{B}$ and in most practically important problems it is impossible to consider these parameters as *a priori* known. To determine them self-consistently, extra equations are needed. These equations are the Maxwell equations, the Ohm equation and the equation of conservation of charges. For practical applications, however, it is more convenient not to use these equations directly, but alternative systems of equations, the form of which depends on the particular problem under consideration. These systems of equations are obviously consistent with the basic equations. One of these problems, particularly relevant to metallurgical applications will be discussed below.

If we assume that the values of the magnetic field at the boundaries are given then the distribution of this field inside the enclosure is determined by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\sigma\mu} \nabla^2 \mathbf{B}, \tag{3}$$

where σ is the electrical conductivity (in $1/(\Omega\ \mathrm{m})$) assumed to be isotropic and constant, μ is the magnetic permeability (in H/m).

Our code will not be designed for the analysis of ferromagnetic materials. Hence, μ will be assumed to be close to the magnetic permeability of the vacuum, $\mu_0 = 1{\cdot}257 \times 10^{-6}$ H/m.

Once the distribution of $\mathbf{B}$ is determined from equation (3), then the distribution of currents can be determined from the equation

$$\mathbf{j} = \frac{1}{\mu} (\nabla \times \mathbf{B}). \tag{4}$$

When deriving (4) we ignored the contribution of the displacement current, which is justified for most practical applications except for processes rapidly varying in time (high frequency waves). These processes, however, are not normally described by conventional CFD codes anyway.

Once we obtained the distribution of **B** and **j** from equations (3) and (4) the distribution of electrical potential, ϕ, and correspondingly the distribution of the electrical field strength, $\mathbf{E} = -\nabla\phi$, can be obtained from the equation:

$$\mathbf{E} = -\nabla\phi = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma}\mathbf{j}. \tag{5}$$

When deriving (5) we ignored the contribution of Hall currents and assumed that Pedersen conductivity is equal to the conductivity along magnetic field lines. These assumptions are justified for metallurgical applications which will be primarily kept in mind.

The values of **B** at the boundaries can be arbitrarily determined. Caution, however, needs to be exercised so that these conditions are not contradictory. In particular, the normal component of the magnetic field at the boundary, B_n should be zero in the case of perfectly conducting walls if we look for the steady state solution to the problem. In this case the tangential components of **j** and **E** as determined by (4) and (5) will be equal to zero as expected.

Explicit expressions of the equations presented in this section in different coordinate systems is given in Appendix. Meanwhile, in the next section we will discuss the analogy between the MHD equations and the conventional CFD equations.

3. ANALOGY BETWEEN MHD AND CONVENTIONAL CFD EQUATIONS

As mentioned in Section 2, the momentum conservation and enthalpy equations in MHD have the same form as in the conventional CFD except the additional terms $\mathbf{j} \times \mathbf{B}$ (Lorentz force) and $\mathbf{j} \cdot \mathbf{E}$ (electrical heating). Hence, conventional CFD solvers can be used for solving these equations in MHD problems by simply adding these terms as source terms.

Equation (3) looks different from the equations used in the conventional CFD. However, even in this case we can see an analogy between this equation and the enthalpy equation solved in conventional CFD, as will be shown below.

In the limit $\sigma \to 0$ equation (3) reduces to:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\sigma\mu}\nabla^2\mathbf{B}, \tag{6}$$

This equation describes the diffusion of the magnetic field and has exactly the same structure as the enthalpy equation with zero velocities and accounting for thermal conductivity effects only: the latter term is mathematically equivalent to $(1/\sigma\mu)\nabla^2\mathbf{B}$ term for all three components of the magnetic field. Hence the solver for the enthalpy equation in the conventional CFD can be used for solving equation (6) by replacing the enthalpy by the components of the magnetic field **B**.

In the opposite limiting case $\sigma \to \infty$ equation (3) reduces to:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{7}$$

This equation (for frozen-in magnetic field lines) cannot obviously be identified with any of the equations used in the conventional CFD as it contains cross dependence between different components of the magnetic field and derivatives with respect to different coordinates for each magnetic field component (see Appendix). However, if we rearrange this equation for one of the components

(x-component in our case) into the following form (cf. equation (3)):

$$\frac{\partial B_x}{\partial t}+\frac{\partial(v_xB_x)}{\partial x}+\frac{\partial(v_yB_x)}{\partial y}+\frac{\partial(v_zB_x)}{\partial z}=\frac{\partial(v_xB_x)}{\partial x}+\frac{\partial(v_xB_y)}{\partial y}+\frac{\partial(v_xB_z)}{\partial z}, \tag{8}$$

it becomes noticeable that the left-hand side of this equation has exactly the same form as the Lagrangian derivative in the enthalpy equation. The terms on the right-hand side of (8) can be treated as the additional 'source terms'. These terms can be treated in a similar way as source terms in the enthalpy equation. Other component of **B** can be treated in exactly the same way (see (A3)–(A5) and (A8)–(A10)).

Returning to the general equation (3) we can now recognize the hidden analogy between this equation and the enthalpy equation. Let us illustrate this by writing equation (3) for the B_x component in Cartesian coordinate system in the form (cf. (A3)):

$$\frac{\partial B_x}{\partial t}+\frac{\partial(v_xB_x)}{\partial x}+\frac{\partial(v_yB_x)}{\partial y}+\frac{\partial(v_xB_x)}{\partial z}=\frac{1}{\sigma\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)B_x+\left[\frac{\partial(v_xB_x)}{\partial x}+\frac{\partial(v_xB_y)}{\partial y}+\frac{\partial(v_xB_z)}{\partial z}\right]. \tag{9}$$

The left-hand side of (9) is equivalent to the Lagrangian derivative of the enthalpy in the enthalpy equation. The first term on the right-hand side is equivalent to the diffusion terms in the enthalpy equation. The term in square brackets in (9) is equivalent to the source term in the enthalpy equation.

The same analogy could be seen for other components of **B** and for different coordinate systems.

Equations (4) and (5) are the explicit equations determining **j** and **E**.

Based on the analogy between MHD and conventional CFD equations discussed in this section one can relatively easily generalize any conventional CFD code so that it can be used for solving MHD problems. In our case we used the finite difference numerical scheme[15] in the FLUENT code to solve the additional magnetic field or electric potential equations. In the next section we briefly describe how this has been done in the case of the FLUENT software package.

4. APPLICATION

The best way to validate the results of numerical computations is to compare them with analytical solutions. In what follows we will do this for the cases of the Poiseuille–Hartmann flow and the shielding of magnetic field oscillations by a conducting medium (skin effect).

4.1. Poiseuille–Hartmann flow

The Poiseuille–Hartmann flow is a 1D flow of conducting and viscous liquid between two stationary walls with external magnetic field imposed perpendicular to these walls. Assuming that the walls are at $y=\pm L$, using the boundary condition that the liquid velocity at $y=\pm L$ is equal to zero, and assuming that the liquid moves in the x-direction under the influence of the pressure gradient, then the liquid velocity between these plates can be determined by the equation[1]:

$$\begin{aligned} v_x &= -\frac{\Gamma G}{\sigma B_0^2}\left(\frac{\cosh\, G-\cosh(Gy/L)}{\sinh\, G}\right),\\ v_y &= v_z = 0, \end{aligned} \tag{10}$$

where $\Gamma=\partial p/\partial x$ is the pressure gradient in the x direction, $B_0=B_y$, $G=\sqrt{\sigma/\eta}B_0L$, η is the coefficient of dynamic viscosity. In the limit $\sigma\to 0$ equation (10) predicts a parabolic velocity profile in a non-magnetized liquid as expected. When $\sigma\neq 0$ then increase of B_0 and σ results in flattening of this profile.

The movement of liquid described by (10) induces the x-component of the magnetic field defined by the equation:[7]

$$B_x = -\frac{B_0 R_m}{G} \frac{\sinh(Gy/L) - (y/L)\sinh G}{\cosh G - 1}, \tag{11}$$

where $R_m = v_x(L=0)L\sigma\mu$ is the magnetic Reynolds number. Note that there is a mistake in the corresponding equation given in Reference 1.

Taking the following values of parameters: density = 1·5 kg/m^3, viscosity = 1·5 · 10^{-4} kg/m/sec, electrical conductivity = 7·14 × 10^5 1/(Ω m), local pressure gradient $\Gamma = -4{\cdot}85$ Pa/m, we computed the functions $v_x(L)$ and $B_x(L)$ from equations (10) and (11) and based on the FLUENT software package with MHD effects accounted for. The results are presented in Figures 1 and 2. As follows from these figures, results predicted by FLUENT are very close to those predicted by (10) and (11). The slight deviations between these results can be explained by a non-zero value of v_y (flow profile slightly deformed along x-direction) which was not accounted for by the analytical solution. This gives us confidence that MHD effects have been correctly implemented into the FLUENT code.

Note that the values of parameters used for our computations might not be very typical for the environment where our code is expected to be applied. The reason for choosing them was that they provide the formation of the asymptotical 1D profile of velocity for realistic values of the distances from the inlet, thus enabling us to compare numerical results with analytical.

4.2. Shielding of magnetic field oscillations

Let us assume that the values of the magnetic field at the plane $y=0$ oscillate with cyclic frequency ν and the medium in the half-plane $y>0$ is stationary and conducting ($0<\sigma<\infty$). In this case equation (3) has an analytical solution in the half-plane $y>0$:[16]

$$\mathbf{B} = \mathbf{B}_0 \exp(-\sqrt{\pi\sigma\mu}y)\cos(\sqrt{\pi\sigma\mu}y - 2\pi\nu t), \tag{12}$$

where t is time, $\mathbf{B}_0$ is the $\mathbf{B}$ vector at $y=0$ and $t=0$.

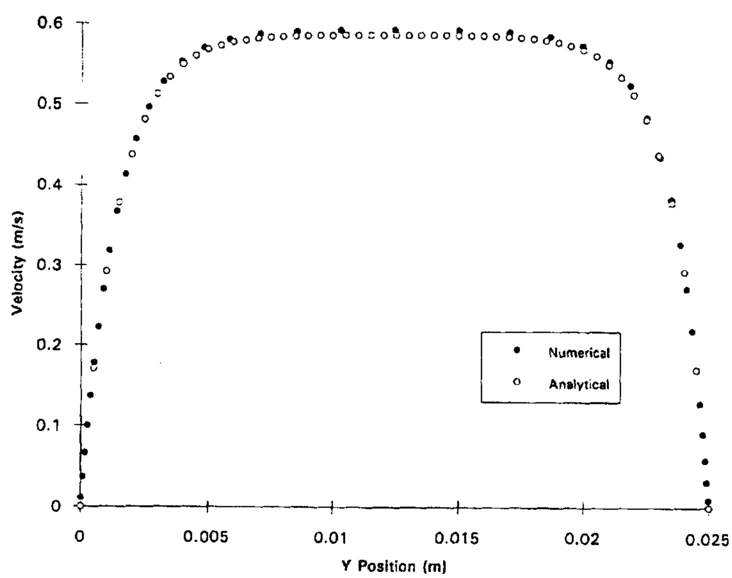

Figure 1. Plots of v_x versus y-position based on analytical (equation (10)) and numerical (FLUENT) results

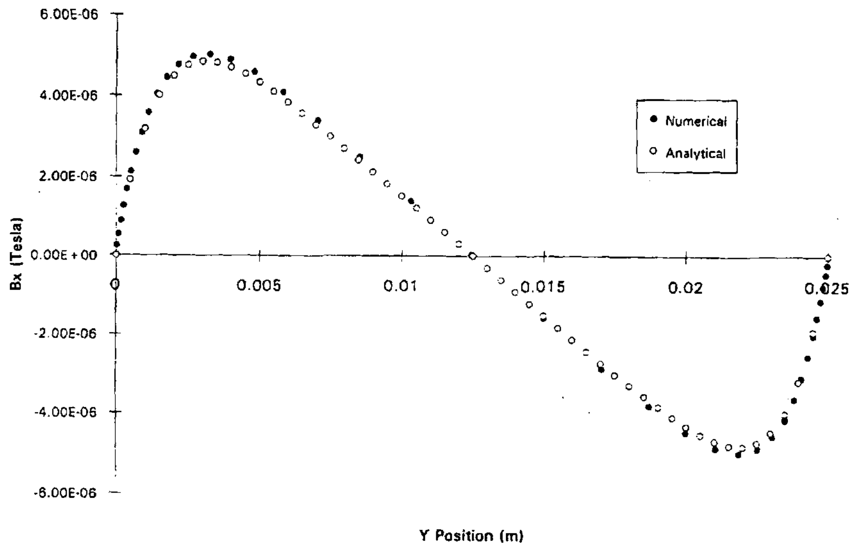

Figure 2. Plots of B_x versus y-position based on analytical (equation (11)) and numerical (FLUENT) results

When deriving (12) we assumed that

$$B(y = 0) = \mathbf{B}_0 \cos(2\pi\nu t).$$

As follows from (12) the amplitude of oscillations decreases with y and the rate of this decrease increases with increasing σ. This effective shielding of magnetic field oscillations by a conducting medium is known as the skin effect.[7]

Taking $\sigma = 7955\ (\Omega\ \mathrm{m})^{-1}$, $\mu = \mu_0$ (this choice of σ and μ provides that $1/\sqrt{\sigma\mu} = 10$ and was made for illustrative purposes only), $\nu = 1\ \mathrm{s}^{-1}$ and $t = 2$ s (this choice of ν and t provides that $2\pi\nu t = 4\pi$; by the end of the second period oscillations practically do not depend on the initial conditions ($t = 0$)) we computed the values of **B** based on our code and validated the numerical solution against predictions of equation (12). As follows from Figure 3 where both analytical and numerical results are presented, agreement between these results seems to be as good as in the case of the Poiseuille–Hartmann flow.

4.3. MHD simulation related to modelling of continuous casting process

Having passed this and a number of other similar tests, our MHD code has been applied to the simulation of continuous casting process in steel making. The efficiency, i.e. cost of the product, is strongly dependant on the casting speed, so that higher casting speed leads to higher productivity. Higher casting speed, however, leads to faster metal flow which prevents the shell from stable growth. Additionally it causes remelting, such as breaking out. Magnetohydrodynamic apparatuses are used in the continuous casting to decelerate and/or stabilize the metal flow. Magnetic stress leads to modifications of the flow so that more favourable conditions for casting processes can be achieved. Numerical simulation based on the code we have described appears to be a powerful device for optimising the configuration and conditions of operation of these magnetohydrodynamic apparatuses. More detailed description of the results of this simulation is beyond the scope of this paper.

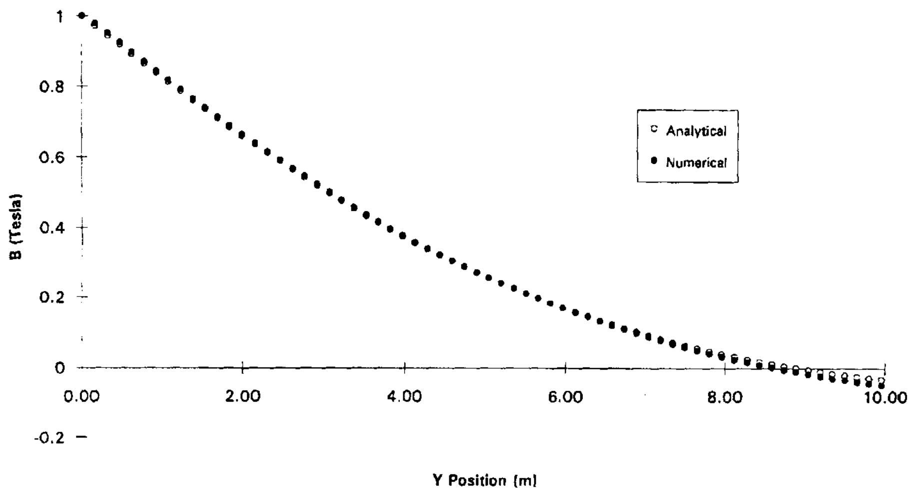

Figure 3. Plots of the modula of $\mathbf{B}(|\mathbf{B}|)$ versus y-position based on analytical (equation (12)) and numerical (FLUENT) results

5. CONCLUSIONS

It is pointed out that from the point of view of numerical computations there exists a similarity between MHD and conventional CFD equations. This allows us to generalize any conventional CFD code so that the effects of MHD can be accounted for. This generalization has actually been made for the FLUENT CFD code. Predictions of the generalized FLUENT code have been validated against the analytical solutions for the Poiseuille–Hartmann flow and shielding of magnetic field oscillations by conducting medium.

ACKNOWLEDGEMENTS

The authors are grateful to K. Kobayashi (Ryutai consultants), H.F. Boysan (Fluent Europe) and S.T. Johansen (SINTEF Materials Technology) for useful discussions and suggestions.

APPENDIX

For practical numerical coding and for discussing the analogy between MHD and CFD equations it is necessary to have explicit expressions of MHD equations in different coordinate systems.

Basic equations in Cartesian coordinates

In the right-handed Cartesian system equations (1)–(5) can be written as:

$$\left.\begin{aligned} \rho\frac{\mathrm{d}v_x}{\mathrm{d}t} &= F_{hx} + (j_yB_z - j_zB_y), \\ \rho\frac{\mathrm{d}v_y}{\mathrm{d}t} &= F_{hy} + (j_zB_x - j_xB_z), \\ \rho\frac{\mathrm{d}v_z}{\mathrm{d}t} &= F_{hz} + (j_xB_y - j_yB_x), \end{aligned}\right\} \tag{A1}$$

$$\frac{d(\rho h)}{dt} = Q_h + (j_x(E_x + (v_yB_z - v_zB_y)) + j_y(E_y + (v_zB_x - v_xB_z)) + j_z(E_z + (v_xB_y - v_yB_x))). \quad \text{(A2)}$$

$$\frac{\partial B_x}{\partial t} = \frac{\partial(v_xB_y - v_yB_x)}{\partial y} - \frac{\partial(v_zB_x - v_xB_z)}{\partial z} + \frac{1}{\sigma\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)B_x, \quad \text{(A3)}$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial(v_yB_z - v_zB_y)}{\partial z} - \frac{\partial(v_xB_y - v_yB_x)}{\partial x} + \frac{1}{\sigma\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)B_y, \quad \text{(A4)}$$

$$\frac{\partial B_z}{\partial t} = \frac{\partial(v_zB_x - v_xB_z)}{\partial x} - \frac{\partial(v_yB_z - v_zB_y)}{\partial y} + \frac{1}{\sigma\mu}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)B_z, \quad \text{(A5)}$$

$$\left.\begin{aligned} j_x &= \frac{1}{\mu}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right), \\ j_y &= \frac{1}{\mu}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right), \\ j_z &= \frac{1}{\mu}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right), \end{aligned}\right\} \quad \text{(A6)}$$

$$\left.\begin{aligned} E_x &= -\frac{\partial\phi}{\partial x} = (v_zB_y - v_yB_z) + \frac{1}{\sigma}j_x, \\ E_y &= -\frac{\partial\phi}{\partial y} = (v_xB_z - v_zB_x) + \frac{1}{\sigma}j_y, \\ E_z &= -\frac{\partial\phi}{\partial z} = (v_yB_x - v_xB_y) + \frac{1}{\sigma}j_z, \end{aligned}\right\} \quad \text{(A7)}$$

Basic equations in general curvilinear coordinates

Equations (1) and (2) in general curvilinear coordinates can be written in the same form as in Cartesian coordinates (see equations (A1) and (A2)) as Cartesian components of the corresponding vectors are stored both in Cartesian and curvilinear coordinates. Presentation of equations (3)–(5) in curvilinear co-ordinates is the following:

$$\begin{aligned} \frac{\partial B_1}{\partial t} &= \frac{1}{J}\sum_i\left[b_2^i\frac{\partial}{\partial\xi_i}(v_1B_2 - v_2B_1) - b_3^i\frac{\partial}{\partial\xi_i}(v_3B_1 - v_1B_3)\right] \\ &\quad + \frac{1}{\sigma\mu J^2}\left[\sum_i b_1^i\frac{\partial}{\partial\xi_i}\left(\sum_{j'}\sum_{i'} b_{j'}^{i'}\frac{\partial B_{j'}}{\partial\xi_{i'}}\right)\right. \\ &\quad \left. + \sum_i b_3^i\frac{\partial}{\partial\xi_i}\sum_{i'}\left(b_3^{i'}\frac{\partial B_1}{\partial\xi_{i'}} - b_1^{i'}\frac{\partial B_3}{\partial\xi_{i'}}\right) - \sum_i b_2^i\frac{\partial}{\partial\xi_i}\sum_{i'}\left(b_1^{i'}\frac{\partial B_2}{\partial\xi_{i'}} - b_2^{i'}\frac{\partial B_1}{\partial\xi_{i'}}\right)\right] \\ &= \frac{1}{J}\sum_i\left[b_2^i\frac{\partial}{\partial\xi_i}(v_1B_2 - v_2B_1) - b_3^i\frac{\partial}{\partial\xi_i}(v_3B_1 - v_1B_3)\right] + \frac{1}{J}\sum_{i,j}\frac{\partial}{\partial\xi_i}Q_{i,j}\frac{\partial B_1}{\partial\xi_j}; \end{aligned} \quad \text{(A8)}$$

$$\frac{\partial B_2}{\partial t} = \frac{1}{J}\sum_i \left[b_3^i \frac{\partial}{\partial \xi_i}(v_2B_3 - v_3B_2) - b_1^i \frac{\partial}{\partial \xi_i}(v_1B_2 - v_2B_1)\right]$$
$$+ \frac{1}{\sigma\mu J^2}\left[\sum_i b_2^i \frac{\partial}{\partial \xi_i}\left(\sum_{j'}\sum_{i'} b_{j'}^{i'}\frac{\partial B_{j'}}{\partial \xi_{i'}}\right)\right.$$
$$\left. + \sum_i b_1^i \frac{\partial}{\partial \xi_i}\sum_{i'}\left(b_1^{i'}\frac{\partial B_2}{\partial \xi_{i'}} - b_2^{i'}\frac{\partial B_1}{\partial \xi_{i'}}\right) - \sum_i b_3^i \frac{\partial}{\partial \xi_i}\sum_{i'}\left(b_2^{i'}\frac{\partial B_3}{\partial \xi_{i'}} - b_3^{i'}\frac{\partial B_2}{\partial \xi_{i'}}\right)\right]$$
$$= \frac{1}{J}\sum_i \left[b_3^i \frac{\partial}{\partial \xi_i}(v_2B_3 - v_3B_2) - b_1^i \frac{\partial}{\partial \xi_i}(v_1B_2 - v_2B_1)\right] + \frac{1}{J}\sum_{i,j}\frac{\partial}{\partial \xi_i} Q_{i,j}\frac{\partial B_2}{\partial \xi_j}; \quad \text{(A9)}$$

$$\frac{\partial B_3}{\partial t} = \frac{1}{J}\sum_i \left[b_3^i \frac{\partial}{\partial \xi_i}(v_3B_1 - v_1B_3) - b_2^i \frac{\partial}{\partial \xi_i}(v_2B_3 - v_3B_2)\right]$$
$$+ \frac{1}{\sigma\mu J^2}\left[\sum_i b_3^i \frac{\partial}{\partial \xi_i}\left(\sum_{j'}\sum_{i'} b_{j'}^{i'}\frac{\partial B_{j'}}{\partial \xi_{i'}}\right)\right.$$
$$\left. + \sum_i b_2^i \frac{\partial}{\partial \xi_i}\sum_{i'}\left(b_2^{i'}\frac{\partial B_3}{\partial \xi_{i'}} - b_3^{i'}\frac{\partial B_2}{\partial \xi_{i'}}\right) - \sum_i b_1^i \frac{\partial}{\partial \xi_i}\sum_{i'}\left(b_3^{i'}\frac{\partial B_1}{\partial \xi_{i'}} - b_1^{i'}\frac{\partial B_3}{\partial \xi_{i'}}\right)\right]$$
$$= \frac{1}{J}\sum_i \left[b_1^i \frac{\partial}{\partial \xi_i}(v_3B_1 - v_1B_3) - b_2^i \frac{\partial}{\partial \xi_i}(v_2B_3 - v_3B_2)\right] + \frac{1}{J}\sum_{i,j}\frac{\partial}{\partial \xi_i} Q_{i,j}\frac{\partial B_3}{\partial \xi_j}. \quad \text{(A10)}$$

Equations (4) and (5) can be presented as:

$$\mathbf{j} = \frac{1}{\mu J}\left\{\left[\left(\sum_i b_2^i\frac{\partial B_3}{\partial \xi_i}\right) - \left(\sum_i b_3^i\frac{\partial B_2}{\partial \xi_i}\right)\right]\mathbf{i}_1 + \left[\left(\sum_i b_3^i\frac{\partial B_1}{\partial \xi_i}\right) - \left(\sum_i b_1^i\frac{\partial B_3}{\partial \xi_i}\right)\right]\mathbf{i}_2\right.$$
$$\left. + \left[\left(\sum_i b_1^i\frac{\partial B_2}{\partial \xi_i}\right) - \left(\sum_i b_2^i\frac{\partial B_1}{\partial \xi_i}\right)\right]\mathbf{i}_3\right\}, \quad \text{(A11)}$$

$$\mathbf{E} = -\nabla\phi = -\frac{1}{J}\sum_j\sum_i\left(b_j^i\frac{\partial \phi}{\partial \xi_i}\right)\mathbf{i}_j$$
$$= \left[(v_3B_2 - v_2B_3) + \frac{1}{\sigma}j_1\right]\mathbf{i}_1 + \left[(v_1B_3 - v_3B_1) + \frac{1}{\sigma}j_2\right]\mathbf{i}_2 + \left[(v_2B_1 - v_1B_2) + \frac{1}{\sigma}j_3\right]\mathbf{i}_3, \quad \text{(A12)}$$

where J is Jacobian, $\mathbf{b}^i = J\nabla\xi_i$ are area vectors, ξ_i are curvilinear coordinates.

REFERENCES

1. D. Bershader and R. Landshoff, 'Magnetohydrodynamics'. In *Handbook of Fluid Dynamics*, ed. by V. L. Streeter, McGraw-Hill Book Company, Inc, 1961, pp. 27-1–27-52.
2. L. E. Kalikhman, *Elements of Magnetogasdynamics*, W. B. Saunders Company, 1967.
3. A. I. Morozov and L. S. Solov'ev, 'Steady-state plasma flow in a magnetic field'. In *Reviews of Plasma Physics*, ed. by M. A. Leontovich, vol. 8, Consultants Bureau, New York–London, 1974, pp. 1–103.
4. K. V. Brushlinskii and A. I. Morozov, 'Calculation of two-dimensional plasma flows in channels'. In *Reviews of Plasma Physics*, ed. by M. A. Leontovich, vol. 8, Consultants Bureau, New York–London, 1974, pp. 105–197.
5. V. F. D'yachenko and V. S. Imshennik, 'Two-dimensional magnetohydrodynamic model for the dense plasma focus of a Z pinch', In *Reviews of Plasma Physics*, ed. by M. A. Leontovich, vol. 8, Consultants Bureau, New York–London, 1974, pp. 199–299.
6. A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko and K. N. Stepanov, *Plasma Electrodynamics*, Pergamon Press, 1977.

7. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.
8. K.-H. Spitzer, M. Dubke and K. Schwerdtfeger, 'Rotational electromagnetic stirring in continuous casting of round strands', *Metallurgical Transactions B*, **37**, 119–131 (1986).
9. S. T. Johansen, 'A general procedure for calculating 3D direct current flows', In *Proceedings of the International Symposium on Electromagnetic Processing of Materials*, Iron and Steel Institute of Japan, Nagoya, 1994, pp. 103–108.
10. S. T Johansen and L. Gu, 'A Reynolds stress model for fluid flow and enthalpy', In *Proceedings of the International Symposium on Electromagnetic Processing of Materials*, Iron and Steel Institute of Japan, Nagoya, 1994, pp. 49–54.
11. J. M. Galpin and Y. Fautrelle, 'Liquid–metal flows induced by low-frequency alternating magnetic fields', *J. Fluid Mechanics*, **239**, 383–408 (1992).
12. T. Sato, T. Ishii and A. Yada, 'Electromagnetic analysis in continuous casting process', *NKK Technical Report*, **143**, 40– 46, 1993 (in Japanese).
13. M. Makhlouf, S. S. Sazhin, C. Leys, D. Toebaert and P. Wild, 'Numerical analysis of the electron energy distribution function in a CO_2 laser discharge', *Infrared Physics*, **34**(5), 525–532 (1993).
14. S. S. Sazhin, P. Wild, E. M. Sazhina, M. Makhlouf, C. Leys and D. Toebaert, 'Three-dimensional modelling of processes in the fast-axial-flow CO_2 laser', *J Physics D: Applied Physics*, **27**, 464–469 (1994).
15. S. V. Patankar, *Numerical Heat Transfer an Fluid Flow*, Hemisphere, Washington, DC, 1980.
16. A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Nauka, Moscow (in Russian), 1972.